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ON HOMOMORPHISMS OF RINGS INTO MATRIX RINGS

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Throughout the present note, all rings and all ring homomorphisms are assumed to be unital. Let R be a ring, and $M_n(R)$ the ring of all $n \times n$ matrices with entries in R . The (i, j) -entry of an element X of $M_n(R)$ will be denoted by $(X)_{ij}$. If $\eta: R \rightarrow S$ is a ring homomorphism, then $M_n(\eta): M_n(R) \rightarrow M_n(S)$ denotes the ring homomorphism defined by $M_n(\eta)(r_{ij}) = (\eta(r_{ij}))$.

Given a ring homomorphism $f: A \rightarrow R$, we put $ax = f(a)x$ and $xa = xf(a)$ for $a \in A$ and $x \in R$, and we say that R is an *A-algebra*, if $R = AR^A$ where $R^A = \{r \in R \mid ar = ra \text{ for all } a \in A\}$. The ring of polynomials in a set of noncommutative variables $X = \{x_i\}$ with coefficients in A is called a *free A-algebra*, and is denoted by $A\langle X \rangle$. An *A-algebra* R is said to be *central* if $R = AR^R$ where R^R is the center of R . The ring of polynomials in a set of commutative variables X with coefficients in A is called a *free central A-algebra*, and is denoted by $A[X]$. A *finitely generated A-algebra* (resp. *finitely generated central A-algebra*) will mean a homomorphic image of $A\langle X \rangle$ (resp. of $A[X]$) for some finite X (see [3]).

The purpose of this note is to prove the following generalizations of [2, Theorem 2] and [1, Theorem 1].

Theorem 1. *Let R be an A-algebra with an A-homomorphism into the $n \times n$ matrix ring over some central A-algebra. Then there is a central A-algebra S and an A-homomorphism $\rho: R \rightarrow M_n(S)$ such that for any A-homomorphism $\sigma: R \rightarrow M_n(T)$, T a central A-algebra, there is a unique A-homomorphism $\eta: S \rightarrow T$ such that $M_n(\eta)\rho = \sigma$.*

Theorem 2. *Let A be a ring with the ascending chain condition on two-sided ideals, and R a finitely generated A-algebra. Then, for each positive integer n , R satisfies the ascending chain condition on ideals P such that R/P can be embedded as A-algebra into the $n \times n$ matrix ring over some central A-algebra.*

The homomorphism ρ in Theorem 1 is called a *universal A-homomorphism* of R .

In advance of proving our theorems, we establish the following

lemmas, whose proofs are heavily due to the technique employed in the proofs of [2, Theorem 1] and [1, Lemma 1].

Lemma 1. *Let $R = A\langle X \rangle$ be the free A -algebra in $X = \{x_\lambda \mid \lambda \in \Lambda\}$, and $S = A[X']$ the free central A -algebra in $X' = \{x_{ij}^\lambda \mid \lambda \in \Lambda, 1 \leq i, j \leq n\}$. Then, the A -homomorphism $\rho: R \rightarrow M_n(S)$ defined by $\rho(x_\lambda) = (x_{ij}^\lambda)$ is a universal A -homomorphism of R .*

Proof. Let T be an arbitrary central A -algebra, and $\sigma: R \rightarrow M_n(T)$ an A -homomorphism. We defined an A -homomorphism $\eta: A[X'] \rightarrow T$ by $\eta(x_{ij}^\lambda) = (\sigma(x_\lambda))_{ij}$. Then it is easy to see that $M_n(\eta)\rho = \sigma$, and that such an η is uniquely determined.

Lemma 2. *Let R be an A -algebra with a universal A -homomorphism $\rho: R \rightarrow M_n(S)$, and I a proper ideal of R . Choose an ideal U of S such that the ideal of $M_n(S)$ generated by $\rho(I)$ is $M_n(U)$.*

(i) *If $U \neq S$, then the A -homomorphism $\bar{\rho}: R/I \rightarrow M_n(S/U)$ induced by ρ is a universal A -homomorphism of R/I .*

(ii) *R/I can be embedded into the $n \times n$ matrix ring over some central A -algebra if and only if $\rho^{-1}(M_n(U)) = I$.*

Proof. (i) Let $\pi: R \rightarrow R/I$ and $\tau: S \rightarrow S/U$ be the canonical maps, and $\sigma: R/I \rightarrow M_n(T)$ (T a central A -algebra) an A -homomorphism. Then we have an A -homomorphism $\eta: S \rightarrow T$ such that $M_n(\eta)\rho = \sigma\pi$. Since $M_n(\eta)\rho(I) = \sigma\pi(I) = 0$, we have $\eta(U) = 0$, and hence there is an A -homomorphism $\bar{\eta}: S/U \rightarrow T$ with $\bar{\eta}\tau = \eta$. It follows then $M_n(\bar{\eta})\bar{\rho}\pi = M_n(\bar{\eta})M_n(\tau)\rho = M_n(\bar{\eta})\tau\rho = M_n(\eta)\rho = \sigma\pi$. Hence $M_n(\bar{\eta})\bar{\rho} = \sigma$, since π is surjective. Now, let $\eta': S/U \rightarrow T$ an arbitrary A -homomorphism with $M_n(\eta')\bar{\rho} = \sigma$. Then we have $M_n(\eta')\bar{\rho} = M_n(\eta')M_n(\tau)\rho = M_n(\eta')\tau\rho = \sigma\pi$. By the uniqueness of η , we obtain $\eta = \eta'\tau = \bar{\eta}\tau$. Hence $\eta' = \bar{\eta}$, since τ is surjective.

(ii) By (i), it is easy to see that R/I can be embedded into the $n \times n$ matrix ring over some central A -algebra if and only if $\bar{\rho}$ is injective, whence it follows our assertion.

Proof of Theorem 1. We may assume that $R = A\langle X \rangle/I$ with some free A -algebra $A\langle X \rangle$ and its ideal I . By Lemma 1, $A\langle X \rangle$ has a universal A -homomorphism $\rho: A\langle X \rangle \rightarrow M_n(S)$. Choose an ideal U of S such that the ideal of $M_n(S)$ generated by $\rho(I)$ is $M_n(U)$. Since $A\langle X \rangle/I$ has an A -homomorphism into the $n \times n$ matrix ring over some central A -algebra, the proof of Lemma 2 (i) enables us to see that $U \neq S$

and the A -homomorphism $\bar{\rho}: A\langle X\rangle/I \rightarrow M_n(S/U)$ induced by ρ is a universal A -homomorphism of $A\langle X\rangle/I$.

Remark. If $\rho: R \rightarrow M_n(S)$ and $\rho': R \rightarrow M_n(S')$ are universal A -homomorphisms of R , then there is an A -isomorphism $\eta: S \rightarrow S'$ such that $\rho' = M_n(\eta)\rho$. Therefore, under the notations of Theorem 1, $\{(\rho(r))_{ij} | r \in R^A, 1 \leq i, j \leq n\}$ generates S as A -algebra. As a consequence, if R is a finitely generated A -algebra, then S is a finitely generated central A -algebra.

Proof of Theorem 2 (cf. also [5, p. 106, Theorem 2.1]). Let $\rho: R \rightarrow M_n(S)$ be a universal A -homomorphism of R , and let $I_1 \subset I_2 \subset \cdots \subset I_k \subset \cdots$ be an ascending chain of ideals of R such that R/I_k can be embedded into the $n \times n$ matrix ring over some central A -algebra. Then we have the following ascending chain of ideals in $M_n(S)$: $\{\rho(I_1)\} \subset \{\rho(I_2)\} \subset \cdots \subset \{\rho(I_k)\} \subset \cdots$, where $\{\rho(I_k)\}$ is the ideal of $M_n(S)$ generated by $\rho(I_k)$. As was noted in the above remark, S is a finitely generated central A -algebra. Hence, $M_n(S)$ satisfies the ascending chain condition on two-sided ideals. Since there exists then a positive integer k such that $\{\rho(I_k)\} = \{\rho(I_{k+1})\} = \cdots$, by Lemma 2 (ii) we obtain $I_k = I_{k+1} = \cdots$.

In conclusion, as application of Theorem 2 together with the following proposition, we shall present several results concerning PI -rings. Recall that a prime PI -ring R has a central simple quotient ring $Q = RK$, where K is the quotient field of the center of R [6, Corollary 1], and that p.i. deg R is the square root of $\dim_K Q$ (see e.g. [4]).

Proposition 1. *If an A -algebra R is a semiprime PI -ring, then R can be embedded into the $n \times n$ matrix ring over some central A -algebra, where n is the least common multiple of p.i. deg R/P for all prime ideals P of R .*

Proof. First, we consider the case that R is prime. Let $Q = RK$ be the central simple quotient ring of R , where K is the quotient field of the center of R . According to $R = AR^A$, we see that $Q = RK = (AR^A)K = (AK)R^A$ is (Artinian) simple. Hence, AK is a prime ring whose center is K , and so by [6, Theorem 2], is a central simple K -algebra. Now, there holds $Q = AK \otimes_K V_Q(AK)$, where the centralizer $V_Q(AK)$ of AK in Q is a central simple algebra. If $L \supset K$ is a splitting field of $V_Q(AK)$, then $V_Q(AK) \otimes_K L \simeq M_m(L)$ and $m \mid \text{p.i. deg } R$. Obviously, $Q \otimes_K L \simeq (AK \otimes_K V_Q(AK)) \otimes_K L \simeq AK \otimes_K M_m(L) \simeq$

$M_m(AK \otimes_K L)$ and $AK \otimes_K L$ is a central A -algebra. Now, we come back to the general case. For every prime ideal P of R , we have seen that R/P can be embedded in the $n \times n$ matrix ring over some central A -algebra S . Hence, we have an embedding $R \rightarrow \prod (R/P) \rightarrow \prod M_n(S) \rightarrow M_n(\prod S)$.

In the following corollaries, we assume that A is a ring with the ascending chain condition on two-sided ideals and that R is a finitely generated A -algebra satisfying a polynomial identity. The former generalizes [7, Lemma 2] and [5, p. 106, Corollary 2.2], and the latter is a generalization of [5, p. 108, Theorem 2.5].

Corollary 1. *R satisfies the ascending chain condition on semiprime ideals.*

Proof. This is immediate by Theorem 2 and Proposition 1.

Corollary 2. *If R is semiprime and S is the set of regular elements of the center of R , then the natural localization R_S is the finite direct sum of finite dimensional central simple algebras.*

Proof. Since R satisfies the ascending chain condition on semiprime ideals (Corollary 1), by [5, p. 108, Corollary 2.4] we have $P_1 \cap \cdots \cap P_t = 0$, where P_1, \dots, P_t are all the minimal prime ideals of R . Now, we can proceed in the same way as that of [5, p. 108, Theorem 2.5] did.

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